J. Math. Anal. Appl. 387 (2012) 645–654

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the structure of the set of equivalent norms on ℓ_1 with the fixed point property ☆

Carlos A. Hernández Linares [a], María A. Japón [a,*], Enrique Llorens-Fuster [b]

[a] Universidad de Sevilla, Departamento de Análisis Matemático, Facultad de Matemáticas, Av. Reina Mercedes, C.P. 41012, Sevilla, Spain
[b] Universidad de Valencia, Departamento de Análisis Matemático, Facultad de Matemáticas, Burjassot, C.P. 46100, Valencia, Spain

ARTICLE INFO

Article history:
Received 10 May 2011
Available online 22 September 2011
Submitted by B. Sims

Keywords:
Nonexpansive mappings
Fixed point theory
Stability
Renorming theory

ABSTRACT

Let $\mathcal{A}$ be the set of all equivalent norms on ℓ_1 which satisfy the FPP. We prove that $\mathcal{A}$ contains rays. In fact, every renorming in ℓ_1 which verifies condition $(*)$ in Theorem 2.1 is the starting point of a (closed or open) ray composed by equivalent norms on ℓ_1 with the FPP. The standard norm $\|\cdot\|_1$ or P.K. Lin's norm defined in Lin (2008) [12] are examples of such norms. Moreover, we study some topological properties of the set $\mathcal{A}$ with respect to some equivalent metrics defined on the set of all norms on ℓ_1 equivalent to $\|\cdot\|_1$.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space and C a subset of X. A mapping $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$. A Banach space is said to satisfy the fixed point property (FPP) if every nonexpansive self-mapping defined on a closed convex bounded subset has a fixed point. It is not difficult to show that the sequence Banach spaces ℓ_1 and c_0, endowed with their natural norms, fail to have the FPP. In fact, the first known Banach spaces with the FPP were the Hilbert spaces [1], the uniformly convex Banach spaces [7] or more generally, the reflexive Banach spaces with normal structure [9]. These results were obtained in 1965 and, for a long time, it was conjectured that every Banach space with the FPP was reflexive. This question was solved by P.K. Lin [12] in 2008 in an unexpected way: Let $\gamma_n = \frac{8^n}{1+8^n}$ for all $n \in \mathbf{N}$ and set $|||x||| := \sup_n \gamma_n \sum_{k=n}^{\infty} |x_n|$ if $x = (x_n)_n \in \ell_1$. Then $|||\cdot|||$ is a renorming on ℓ_1 which has the FPP. Therefore, $(\ell_1, |||\cdot|||)$ is a nonreflexive Banach space satisfying the FPP. Since P.K. Lin's result, more nonreflexive Banach spaces with the FPP have been found (see [5,6,8]).

This paper is mainly divided in four sections. In Section 2 we will obtain new families of renormings on ℓ_1 satisfying the FPP. In fact, in a subsequent paper, P.K. Lin [13] established four conditions which are sufficient to assure that a renorming on ℓ_1 verifies the FPP. We will check that many of the norms obtained in our main result do not satisfy P.K. Lin's condition.

In Section 3 we will study whether P.K. Lin's norm produces some stability of the fixed point property. It is known that the fixed point property is not preserved under isomorphisms: $(\ell_1, \|\cdot\|_1)$ fails the FPP whereas $(\ell_1, |||\cdot|||)$ does have the FPP. However for many classical reflexive Banach spaces, the fixed point property is preserved under isomorphisms whenever the Banach–Mazur distance between the original space and the isomorphic one is less than a certain constant. This is known as the problem of the stability of the fixed point property and it can be formulated as follows: let X be a Banach space

☆ The first two authors are partially supported by DGES, Grant MTM 2009-10696-C02-01 and Junta de Andalucía, Grant FQM-127. The third author has been partially supported by DGES, Grant MTM 2009-10696-C02-02.
* Corresponding author.
E-mail addresses: carloshl@us.es (C.A. Hernández Linares), japon@us.es (M.A. Japón), enrique.llorens@uv.es (E. Llorens-Fuster).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.09.029

with the FPP and Y a Banach space isomorphic to X. Does there exist some constant $K = K(X) > 1$ such that Y has the FPP whenever $d(X,Y) < K$? This problem has been widely studied by many researchers and many geometric properties have turned out to be useful to determine an upper bound for the Banach–Mazur distance which assures the transmission of the FPP (see Chapter 7 in [10] and the references therein for a broad exposition about this topic). We will prove that P.K. Lin's norm, along with most of the renormings of ℓ_1 with the FPP, fail to produce stability of the FPP. This fact contrasts with the case of the classical norms in reflexive Banach spaces with the FPP.

Finally, in Section 4 we consider the convex cone $\mathcal{P}$ of all norms on ℓ_1 which are equivalent to $\|\cdot\|_1$ and its subset $\mathcal{A}$ given by the norms of $\mathcal{P}$ satisfying the FPP. We deduce that $\mathcal{A}$ contains rays and we study some properties concerning to the structure of the sets $\mathcal{A}$ and $\mathcal{P} \setminus \mathcal{A}$.

2. Family of norms on ℓ_1 with the FPP

Let X be a Banach space, C a subset of X and $T : C \to C$ a mapping. A sequence (x_n) in C is called an approximate fixed point sequence (a.f.p.s.) if

$$\lim_n \|x_n - Tx_n\| = 0.$$

It is not difficult to prove, by using Banach's Contraction Principle, that every nonexpansive mapping has an approximate fixed point sequence $(x_n) \subset C$ if C is a convex, bounded, closed subset of X.

We will introduce some notation which will be used throughout the paper: Let (γ_k) be any non-decreasing sequence in $(0,1)$ converging to 1 and denote by $|||\cdot|||$ the renorming on ℓ_1 given by

$$|||x||| := \sup_k \gamma_k \sum_{n=k}^{\infty} |x_n|.$$

Denote $R_k(x) := \sum_{n=k}^{\infty} |x_n|$ for all $x = (x_n)_n \in \ell_1$.

The above norm is a generalization of the original norm given by P.K. Lin in [12] and it has been proved that $(\ell_1, |||\cdot|||)$ has the FPP for any sequence (γ_k) in the above conditions (see [8] or Example 1 in [13]). The usual norm on ℓ_1 will be denoted by $\|\cdot\|_1$ and P_k denotes the natural projection on ℓ_1 for every $k \in \mathbf{N}$.

The main result of this section is the following:

Theorem 2.1. *Let $p(\cdot)$ be an equivalent norm to the usual norm on ℓ_1 such that*

$$\limsup_n p(x_n + x) = \limsup_n p(x_n) + p(x) \tag{$*$}$$

for every w^*-null sequence (x_n) and for all $x \in \ell_1$. Then the norm

$$|\cdot|_p = p(\cdot) + \lambda |||\cdot|||$$

has the FPP for every $\lambda > 0$.

Before starting with the proof, we state a general result for Banach spaces which fail to satisfy the FPP and that can be deduced from the existence of approximate fixed point sequences and Cantor's Intersection Theorem (see Lemma 1 and the subsequent remark in [8] for a proof):

Lemma 2.2. *Let $(X, \|\cdot\|)$ be a Banach space with a linear topology τ such that every bounded sequence has a τ-convergent subsequence. Let C be a closed convex bounded subset of X and $T : C \to C$ a nonexpansive mapping. If T is fixed point free, then there exists a closed convex T-invariant subset D of C such that*

$$\inf\Big\{\limsup_n \|x_n - x\|\colon\ (x_n) \subset D,\ (x_n)\ \text{a.f.p.s.},\ x_n \to x \text{ in } \tau\Big\} > 0.$$

Now we proceed to prove Theorem 2.1. The proof is an adaptation of Lin's original proof in [12]. Notice that

$$|x|_p := \sup_k \rho_k(x),$$

where $\rho_k(x) := p(x) + \lambda\gamma_k R_k(x)$. It is not difficult to check that for all weak*-null sequence and for all $x \in \ell_1$ the following property holds:

$$\limsup_n \rho_k(x_n + x) = \limsup_n \rho_k(x_n) + \rho_k(x) \tag{I}$$

for all $k \in \mathbf{N}$. Moreover, $\limsup_n R_k(x_n) = \limsup_n |||x_n|||$ for all $k \in \mathbf{N}$ if (x_n) is a weak*-null sequence in ℓ_1.

Proof of Theorem 2.1. Assume to the contrary that $(\ell_1, |\cdot|_p)$ fails the FPP. Let T and D be as in Lemma 2.2 and τ the $\sigma(\ell_1, c_0)$-topology in ℓ_1, which will be denoted by $w^\star$.

Define

$$s := \inf\Big\{\limsup_n |x_n - x|_p\colon (x_n) \subset D,\ (x_n) \text{ a.f.p.s.},\ x_n \xrightarrow{w^*} x\Big\}$$

which is strictly positive. Consider the set $A(D)$ of all $(x_n) \subset D$, such that (x_n) is an a.f.p.s. converging $w^\star$ to some $x \in \ell_1$ and such that $\lim_n p(x_n - w)$, $\lim_n |||x_n - w|||$ and $\lim_n R_k(x_n - w)$ exist for all $w \in \ell_1$ and for all $k \in \mathbf{N}$. Notice that, from the separability of ℓ_1, one can infer that

$$s = \inf\Big\{\lim_n |x_n - x|_p\colon (x_n) \in A(D),\ x_n \xrightarrow{w^*} x\Big\}.$$

Without loss of generality we can assume that $s = 1$. Define

$$c := \inf\big\{|||x|||\colon p(x) = \lambda\big\}; \qquad a := \inf\big\{|||x|||\colon |x|_p = \lambda\big\},$$

which are constants strictly greater than zero since the involved norms are equivalent.

We choose $\varepsilon_1 > 0$ such that

$$\frac{1+\varepsilon_1}{1+c} + 2\varepsilon_1 < 1$$

and an a.f.p.s. $(x_n) \in A(D)$ such that weak*-$\lim_n x_n = x$ and $\lim_n |x_n - x|_p < 1 + \varepsilon_1$. By translation, we can suppose that $x = 0$.

Define

$$K := \Big\{z \in D\colon \lim_n |x_n - z|_p \leqslant 2 + 2\varepsilon_1\Big\}.$$

Notice that K is nonempty, closed, convex, bounded and T-invariant. In fact, there exists $n_0 \in \mathbf{N}$ such that $x_n \in K$ for $n \geqslant n_0$.

Define r by

$$r := \inf\Big\{\lim_n |y_n - y|_p\colon (y_n) \subset K,\ (y_n) \in A(D),\ y_n \xrightarrow{w^*} y\Big\}.$$

From the definition of r we have

$$1 \leqslant r \leqslant \lim_n |x_n|_p < 1 + \varepsilon_1. \tag{1}$$

Define $A(K) = \{(y_n) \in A(D)\colon y_n \in K, \forall n \in \mathbf{N}\}$ and consider $(y_n) \in A(K)$ an arbitrary a.f.p.s. such that $y_n \xrightarrow{w^*} y$. Then for all $k \in \mathbf{N}$ we have

$$\begin{aligned}
2 + 2\varepsilon_1 &\geqslant \limsup_m \lim_n |x_n - y_m|_p \\
&\geqslant \limsup_m \lim_n \rho_k(x_n - y_m) \\
&= \limsup_m \Big[\limsup_n \rho_k(x_n) + \rho_k(y_m)\Big] \quad \big(\text{by (I)}\big) \\
&= \limsup_n \rho_k(x_n) + \limsup_m \rho_k(y_m - y) + \rho_k(y) \quad \big(\text{by (I)}\big) \\
&= \lim_n p(x_n) + \lambda\gamma_k \lim_n R_k(x_n) + \lim_m p(y_m - y) + \lambda\gamma_k \lim_n R_k(y_m - y) + \rho_k(y) \\
&= \lim_n p(x_n) + \lambda\gamma_k \lim_n |||x_n||| + \lim_m p(y_m - y) + \lambda\gamma_k \lim_n |||y_m - y||| + \rho_k(y) \\
&\geqslant \gamma_k\Big[\lim_n |x_n|_p + \lim_m |y_m - y|_p\Big] + \rho_k(y) \\
&\geqslant 2\gamma_k + \rho_k(y).
\end{aligned}$$

Hence, if $(y_n) \in A(K)$ is an a.f.p.s. and weak*-$\lim y_n = y$, we have

$$\begin{aligned}
\rho_k(y) &\leqslant 2(1 - \gamma_k) + 2\varepsilon_1 &(2)\\
&< 2 + 2\varepsilon_1. &(3)
\end{aligned}$$

Choose m such that

$$\frac{1+\varepsilon_1}{1+c} + 2\varepsilon_1 < m < 1.$$

Notice that $p(w) \leqslant |w|_p/(1+c)$ for all $w \in \ell_1$, by definition of the constant c. Therefore

$$\lim_n p(x_n) \leqslant \frac{\lim_n |x_n|_p}{1+c} < \frac{1+\varepsilon_1}{1+c}$$

and there exists $n_0 \in \mathbf{N}$ such that

$$p(x_{n_0}) < \frac{\lim_n |x_n|_p}{1+c} < \frac{1+\varepsilon_1}{1+c}.$$

On the other hand,

$$\limsup_k \rho_k(x_{n_0}) = p(x_{n_0}) + \lambda \limsup_k \gamma_k R_k(x_{n_0}) = p(x_{n_0}),$$

so we can find $k_0 \in \mathbf{N}$ such that the following hold:

$$\rho_k(x_{n_0}) < \frac{1+\varepsilon_1}{1+c} \tag{4}$$

and

$$q_k := \frac{1+\varepsilon_1}{1+c} + 2(1-\gamma_k) + 2\varepsilon_1 < m < 1 \leqslant r \tag{5}$$

for all $k \geqslant k_0$.

Since the set K is bounded there exists some $H > 0$ such that $|x|_p < H$ for all $x \in K$. Hence

$$\rho_k(x_{n_0}) < H \quad \text{for all } k \in \mathbf{N}. \tag{6}$$

Define $m_0 := 1 - a(1-\gamma_{k_0})$ which is strictly less than 1 and

$$h := H + 2 + 2\varepsilon_1 > r > m_0 r \quad \bigl(\text{from (1)}\bigr). \tag{7}$$

We take $\beta \in (0,1)$ such that

$$\beta < \frac{2r(1-m_0)}{h - m_0 r}.$$

Note that

$$(2-\beta)r + \beta m = 2r - \beta(r-m) < 2r$$

and

$$(2-\beta)m_0 r + \beta h = 2m_0 r + \beta(h - m_0 r) < 2m_0 r + 2r(1-m_0) = 2r.$$

Therefore, we can find $\varepsilon_2 > 0$ such that

$$(2-\beta)(r+\varepsilon_2) + \beta m < 2r, \tag{8}$$

$$(2-\beta)m_0(r+\varepsilon_2) + \beta h < 2r. \tag{9}$$

Set

$$M := \max\bigl\{(2-\beta)(r+\varepsilon_2) + \beta m, (2-\beta)m_0(r+\varepsilon_2) + \beta h\bigr\}$$

which is strictly less than $2r$.

Take $(y_n) \in A(K)$ an a.f.p.s. such that weak*-$\lim_n y_n = y$ and $\lim_n |y_n - y|_p < r + \varepsilon_2$. Consider $N_0 \in \mathbf{N}$ such that

$$|y_n - y|_p < r + \varepsilon_2 \tag{10}$$

for all $n \geqslant N_0$.

Moreover,

$$\begin{aligned}
\lim_n \rho_k(y_n - y) &= \lim_n p(y_n - y) + \lambda\gamma_k \lim_n R_k(y_n - y) \\
&= \lim_n p(y_n - y) + \lambda\gamma_k \lim_n |||y_n - y||| \\
&= \lim_n |y_n - y|_p - (1-\gamma_k)\lambda \lim_n |||y_n - y||| \\
&\leqslant \bigl[1 - (1-\gamma_k)a\bigr] \lim_n |y_n - y|_p \quad \text{by definition of } a \\
&< \bigl[1 - (1-\gamma_k)a\bigr](r + \varepsilon_2),
\end{aligned}$$

and we can find $N_1 \geqslant N_0$ such that

$$\rho_k(y_n - y) < \big[1 - (1-\gamma_k)a\big](r+\varepsilon_2) \leqslant m_0(r+\varepsilon_2) \tag{11}$$

for all $n \geqslant N_1$ and $k = 1, \ldots, k_0$.

Define now

$$z_0 := \beta x_{n_0} + (1-\beta) y_{N_1}$$

which belongs to K, since K is convex.

Let us now prove that $\lim_n |y_n - z_0|_p \leqslant M$. In order to do this, we will check that for all $k \in \mathbf{N}$ and $n \geqslant N_1$ we have

$$\rho_k(y_n - z_0) \leqslant M.$$

Notice that

$$y_n - z_0 = y_n - y - (1-\beta)(y_{N_1} - y) - \beta(x_{n_0} - y).$$

Fix $n \geqslant N_1$. We split the proof into two cases:

Case 1: $k > k_0$:

$$\begin{aligned}
\rho_k(y_n - z_0) &\leqslant \rho_k(y_n - y) + (1-\beta)\rho_k(y_{N_1} - y) + \beta\big[\rho_k(x_{n_0}) + \rho_k(y)\big] \\
&\leqslant |y_n - y|_p + (1-\beta)|y_{N_1} - y|_p + \beta\big[\rho_k(x_{n_0}) + \rho_k(y)\big] \quad \text{by definition of } |\cdot|_p \\
&< (2-\beta)(r+\varepsilon_2) + \beta\big[\rho_k(x_{n_0}) + \rho_k(y)\big] \quad \big(\text{by (10)}\big) \\
&< (2-\beta)(r+\varepsilon_2) + \beta\left[\frac{1+\varepsilon_1}{1+c} + 2(1-\gamma_k) + 2\varepsilon_1\right] \quad \big(\text{by (4) and (2)}\big) \\
&= (2-\beta)(r+\varepsilon_2) + \beta q_k < (2-\beta)(r+\varepsilon_2) + \beta m \quad \big(\text{from (5)}\big) \\
&< 2r \quad \big(\text{from (8)}\big).
\end{aligned}$$

Case 2: $k \leqslant k_0$:

$$\begin{aligned}
\rho_k(y_n - z_0) &\leqslant \rho_k(y_n - y) + (1-\beta)\rho_k(y_{N_1} - y) + \beta\big[\rho_k(x_{n_0}) + \rho_k(y)\big] \\
&\leqslant m_0(2-\beta)(r+\varepsilon_2) + \beta\big[\rho_k(x_{n_0}) + \rho_k(y)\big] \quad \big(\text{from (11)}\big) \\
&< m_0(2-\beta)(r+\varepsilon_2) + \beta[H + 2 + 2\varepsilon_1] \quad \big(\text{by (6) and (3)}\big) \\
&\leqslant m_0(2-\beta)(r+\varepsilon_2) + \beta h \quad \big(\text{by (7)}\big) \\
&< 2r \quad \big(\text{by (9)}\big).
\end{aligned}$$

Then $\rho_k(y_n - z_0) \leqslant M$ for all $k \in \mathbf{N}$ and for all $n \geqslant N_1$. So, $|y_n - z_0|_p \leqslant M$ for all $n \geqslant N_1$ and

$$\limsup_n |y_n - z_0|_p \leqslant M.$$

Define now

$$K_0 := \Big\{z \in K\colon \limsup_n |y_n - z|_p \leqslant M\Big\}.$$

Notice that K_0 is nonempty since $z_0 \in K_0$ and we can find $(w_m) \in A(D)$, $(w_m) \subset K_0$ such that weak*-$\lim_m w_m = w \in \ell_1$. Now, for all $k \in \mathbf{N}$:

$$\begin{aligned}
M &\geqslant \limsup_m \limsup_n |y_n - w_m|_p \\
&\geqslant \limsup_m \limsup_n \rho_k(y_n - w_m) \\
&= \lim_n \rho_k(y_n - y) + \lim_m \rho_k(w_m - w) + \rho_k(y - w) \\
&\geqslant \lim_n p(y_n - y) + \lambda\gamma_k \lim_n |||y_n - y||| + \lim_m p(w_m - w) + \lambda\gamma_k \lim_m |||w_m - w|||.
\end{aligned}$$

Taking limits when k goes to infinity

$$M \geqslant \lim_n |y_n - y|_p + \lim_m |w_m - w|_p \geqslant r + r = 2r$$

which contradicts the definition of M. Therefore $(\ell_1, |\cdot|_p)$ has the FPP as we wanted to prove. $\square$

If X is a Banach space with a basic sequence (e_n), recall that a sequence of non-zero vectors (u_j) in X is called a block basic sequence of (e_n) if $u_j = \sum_{i=p_j+1}^{p_{j+1}} a_j e_j$ with scalars a_j and $p_1 < p_2 < \cdots$.

At this point, we should note that P.K. Lin proved the following result in [13]:

Let $|\cdot|$ be an equivalent norm on ℓ_1 satisfying the following four properties:

(1) there are $\alpha > 4$ and a positive (decreasing) sequence (α_n) in $(0,1)$ such that for any normalized block basis $\{f_n\}$ of $(\ell_1, |\cdot|)$ and $x \in \ell_1$ with $P_{k-1}(x) = x$ and $|x| < \alpha_k$,

$$\limsup_n |f_n + x| \leqslant 1 + \frac{|x|}{\alpha},$$

(2) there are two strictly decreasing sequences $\{\beta_k\}$ and $\{\nu_k\}$ with $\lim_k \beta_k = 0$ and $\lim_k \nu_k = 1$, such that for any normalized block basis $\{f_n\}$ of $(\ell_1, |\cdot|)$ and x with $(I - P_k)(x) = x$,

$$\liminf_n |f_n + x| \geqslant 1 - \beta_k + \nu_k^{-1}|x|,$$

(3) for any $k \in \mathbb{N}$, $\|I - P_k\| = 1$, and
(4) the unit ball of $(\ell_1, |\cdot|)$ is $\sigma(\ell_1, c_0)$-closed.

Then $(\ell_1, |\cdot|)$ has the FPP.

Notice that P.K. Lin's norm $|||\cdot|||$ satisfies the above four conditions. On the other hand, it is obvious that $\|\cdot\|_1$ cannot satisfy P.K. Lin's properties, but it is easy to check that $\|\cdot\|_1$ satisfies conditions (2), (3) and (4) stated above and it only fails condition (1). Therefore, condition (1) turns out to be the key for the author in [13] to prove the FPP. A natural question is if there are equivalent norms on ℓ_1 having the FPP and failing the sufficient conditions given in [13].

Let us check that the equivalent norm $|\cdot| = \|\cdot\|_1 + |||\cdot|||$ does not verify property (1) from [13]:

Consider the sequence $f_n = \frac{e_n}{1+\gamma_n}$ which is a normalized block basis of $(\ell_1, |\cdot|)$ and let (α_n) be any sequence in $(0,1)$. Let $0 < t_k < \alpha_k$ and $x = \frac{t_k}{1+\gamma_1} e_1$ which verifies $P_{k-1}(x) = x$ for all $k \geqslant 2$ and $|x| = t_k < \alpha_k$. Notice that

$$|f_n + x| = \frac{1}{1+\gamma_n} + \frac{t_k}{1+\gamma_1} + \max\left\{\gamma_1\left(\frac{t_k}{1+\gamma_1} + \frac{1}{1+\gamma_n}\right), \gamma_n \frac{1}{1+\gamma_n}\right\}.$$

Assume that there exists some $\alpha > 4$ verifying property (1). Then

$$\begin{aligned} 1 + \frac{t_k}{\alpha} = 1 + \frac{|x|}{\alpha} &\geqslant \limsup_n |f_n + x| \\ &\geqslant \limsup_n \left[\frac{1}{1+\gamma_n} + \frac{t_k}{1+\gamma_1} + \gamma_n \frac{1}{1+\gamma_n}\right] = 1 + \frac{t_k}{1+\gamma_1} \end{aligned}$$

which implies that $4 < \alpha \leqslant 1 + \gamma_1$, which is a contradiction. In fact, if we want $\|\cdot\|_1 + |||\cdot|||$ to verify a property similar to (1) then α would have to be less or equal to 1 (since γ_1 can be any number in $(0,1)$), but $\alpha = 1$ in (1) is not valid for establishing sufficient conditions, since $\|\cdot\|_1$ would satisfy (1) with $\alpha = 1$.

Therefore, $\|\cdot\|_1 + |||\cdot|||$ is an equivalent norm failing to satisfy P.K. Lin's sufficient conditions to ensure the FPP. On the other hand, from Theorem 2.1 we can deduce that for every linear combination

$$\mu\|x\|_1 + \lambda|||x|||; \quad \mu \geqslant 0,\ \lambda > 0$$

the space ℓ_1 endowed with this norm has the FPP.

But Theorem 2.1 can be applied to more general norms on ℓ_1. For instance, every equivalent norm $p(\cdot)$ on ℓ_1 which separates disjoint vectors (i.e., $p(x+y) = p(x) + p(y)$ if $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$) satisfies the hypotheses of Theorem 2.1. One example of such norm can be

$$\|x\|_\rho := \sum_{n=1}^{\infty} r_n |x_n|$$

if $x = (x_n)_n \in \ell_1$, where $\rho = (r_n)$ is any bounded sequence on $\mathbf{R}$ such that $\inf_n r_n > 0$. Then $\|\cdot\|_\rho$ is an equivalent norm to $\|\cdot\|_1$, fails to have the FPP (since $(\ell_1, \|\cdot\|_\rho)$ is isometric to $(\ell_1, \|\cdot\|_1)$ via the isometry $S(x) = (r_1x_1, r_2x_2, \ldots)$) and

$$\mu\|x\|_\rho + \lambda|||x|||$$

has the FPP for every $\mu \geqslant 0$ and $\lambda > 0$.

More general: let $k \in \mathbf{N}$. Let $\|\cdot\|_k$ be any norm on $\mathbf{R}^k$ and $p(\cdot)$ be a norm on ℓ_1 which separates disjoint vectors. Then

$$|x| = \alpha_1 p\big((I - P_k)x\big) + \alpha_2\|P_k x\|_k$$

satisfies condition $(*)$ in Theorem 2.1 for all $\alpha_1, \alpha_2 > 0$. Here we can include the norms

$$|x| = \alpha_1 \sum_{n=k+1}^{\infty} |x_n| + \alpha_2 \left(\sum_{n=1}^{k} |x_n|^q \right)^{1/q}$$

where $k \in \mathbf{N}$ and $1 \leqslant q < +\infty$.

3. Stability and fixed point property

As we explained in the Introduction section, the FPP is not preserved by isomorphisms. However, it is known that for many classical reflexive Banach spaces, the fixed point property is inherited by "close" renormings in the sense of the Banach–Mazur distance. Recall that if X and Y are Banach spaces, the Banach–Mazur distance is defined by

$$d(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\|: \ T \text{ isomorphism between } X \text{ and } Y\}.$$

For instance, if $X = \ell_p$, $1 < p < +\infty$, endowed with its original norm, and Y is isomorphic to ℓ_p, it is known that Y has the FPP if $d(X, Y) < (1 + 2^{\frac{1}{p-1}})^{\frac{p-1}{p}}$ [3]. When X is uniformly convex and $d(X, Y) \leqslant WCS(X)$, the Banach space Y has the FPP [2]. If H is a Hilbert space and $d(H, Y) < \sqrt{\frac{5+\sqrt{17}}{2}}$ [14] then Y has the FPP.

Now we go back to the space ℓ_1: fix as usual any $(\gamma_n) \subset (0, 1)$ non-decreasing with $\lim_n \gamma_n = 1$ and consider the corresponding $|||\cdot|||$-norm defined by

$$|||x||| = \sup_k \gamma_k \sum_{n=k}^{\infty} |x_n|.$$

Does $(\ell_1, |||\cdot|||)$ produce some stability? That is, does there exist some constant $K > 1$ such that Y has the FPP whenever $d((\ell_1, |||\cdot|||), Y) < K$? By means of the following example we will check that the answer to this question is that such constant $K > 1$ cannot exist in sharp contrast with the stability bounds that are known for classical reflexive Banach spaces.

Example 3.1. We will check that for every $K > 1$ there is an equivalent norm on ℓ_1, whose Banach–Mazur distance to $(\ell_1, |||\cdot|||)$ is less than K but failing the FPP.

Take k_0 such that $1/\gamma_{k_0} < K$ and define the norm

$$|||x|||_{k_0} = \sup_{1 \leqslant k \leqslant k_0} \gamma_k \sum_{n=k}^{\infty} |x_n|; \quad \text{for all } x = (x_n)_n \in \ell_1.$$

It is not difficult to check that $|||x|||_{k_0} \leqslant |||x||| \leqslant \frac{1}{\gamma_{k_0}} |||x|||_{k_0}$ for all $x \in \ell_1$ and therefore

$$d\big((\ell_1, |||\cdot|||), (\ell_1, |||\cdot|||_{k_0})\big) \leqslant \frac{1}{\gamma_{k_0}} < K.$$

Define the linear mapping $T : (\ell_1, \|\cdot\|_1) \to (\ell_1, |||\cdot|||_{k_0})$ given by

$$Tx = \frac{1}{\gamma_{k_0}} (\underbrace{0, \ldots, 0}_{k_0 - 1}, x_1, x_2, \ldots) \quad \text{for all } x = (x_n)_n \in \ell_1.$$

Since T is an isometry the space $(\ell_1, |||\cdot|||_{k_0})$ contains an isometric copy of ℓ_1 and hence it fails the FPP.

Notice that, using the above example, we can check that none of the norms $\|x\|_1 + \lambda|||x|||$ have stability. Indeed, fix $\lambda > 0$ and consider the linear mapping $S : (\ell_1, \|\cdot\|_1) \to (\ell_1, \|\cdot\|_1 + \lambda|||\cdot|||_{k_0})$ given by

$$S(x) = \frac{1}{1 + \lambda\gamma_{k_0}} (\underbrace{0, \ldots, 0}_{k-1}, x_1, x_2, \ldots) \quad \text{for all } x = (x_n)_n \in \ell_1.$$

This mapping is again an isometry which shows that $(\ell_1, \|\cdot\|_1 + \lambda|||\cdot|||_{k_0})$ fails the FPP and we can easily check that $d((\ell_1, \|\cdot\|_1 + \lambda|||\cdot|||), (\ell_1, \|\cdot\|_1 + \lambda|||\cdot|||_{k_0})) \leqslant \frac{1}{\gamma_{k_0}}$. Since we can choose γ_{k_0} as close as 1 as we want, we deduce that neither of these norms have stability for the fixed point property.

Notice that from the above ideas one can prove that none of the norms given in the examples of [13] with the FPP have stability. Therefore, a natural question is the following: Does there exist an equivalent norm $p(\cdot)$ on ℓ_1 such that $(\ell_1, p(\cdot))$ has the FPP and $p(\cdot)$ produces stability of the FPP?

We finish this section by noticing that Theorem 2.1 also shows the following: For any norm $p(\cdot)$ satisfying condition $(*)$ but failing the FPP, we may find an equivalent norm as close as we like, in the Banach–Mazur sense, which verifies the FPP. So the property of failing the FPP is not stable for norms satisfying condition $(*)$ either.

4. On the structure of the set of equivalent norms on ℓ_1 with the FPP

Let $\mathcal{P}$ be the set of all norms on ℓ_1 equivalent to $\|\cdot\|_1$ and define

$$\mathcal{A} := \{p(\cdot):\ p \in \mathcal{P} \text{ and } (\ell_1, p(\cdot)) \text{ has the FPP}\}.$$

In the set $\mathcal{P}$ we can define a natural metric given by

$$h(p,q) := H(B_p, B_q)$$

where H is the Hausdorff distance and B_p, B_q are the unit balls of (ℓ_1, p), (ℓ_1, q) respectively. It is not difficult to check that $(\mathcal{P}, h)$ is a complete metric space.

Another metric defined on $\mathcal{P}$ is the following

$$\rho(p,q) = \sup\{|p(x) - q(x)|:\ \|x\|_1 \leqslant 1\}.$$

Then $\mathcal{P}$ is an open set of the complete metric space $(\mathcal{Q}, \rho)$ of all continuous seminorms on $(\ell_1, \|\cdot\|_1)$ endowed with the metric ρ defined as above. Moreover, the metric spaces $(\mathcal{P}, h)$ and $(\mathcal{P}, \rho)$ are equivalent [4].

The fixed point property is an isometric property which implies that $\lambda p \in \mathcal{A}$ (for $\lambda > 0$) whenever $p \in \mathcal{A}$. In this sense, we could restrict the study of the equivalent norms verifying the FPP to the set of normalized norms, that is, equivalent norms which satisfy $\sup_{\|x\|_1 \leqslant 1} p(x) = 1$. Denote by $\mathcal{E}$ the set of all equivalent normalized norms on ℓ_1. Following the idea of the Banach–Mazur distance we can also define a metric on $\mathcal{E}$ by

$$d(p,q) = \log \frac{b_{p,q}}{a_{p,q}} = \log \|i\| \left\|i^{-1}\right\|$$

where i is the identity operator and

$$a_{p,q} := \inf\left\{\frac{p(x)}{q(x)}:\ \|x\|_1 = 1\right\}; \qquad b_{p,q} := \sup\left\{\frac{p(x)}{q(x)}:\ \|x\|_1 = 1\right\}.$$

The space $(\mathcal{E}, d)$ is metric and it can be proved that $(\mathcal{E}, \rho)$, $(\mathcal{E}, h)$, $(\mathcal{E}, d)$ are equivalent metric spaces [15] so the three metrics generate the some topology on $\mathcal{E}$.

Until 2008 it was conjectured that the set $\mathcal{A}$ was empty. In that year, P.K. Lin proved that $|||\cdot||| \in \mathcal{A}$ [12], where $|||\cdot|||$ is the norm introduced in Section 2. It would be interesting to know some properties of the set $\mathcal{A}$.

Notice that $\gamma_1\|x\|_1 \leqslant |||x||| \leqslant \|x\|_1$ for all $x \in \ell_1$, which implies that

$$\rho(|||\cdot|||, \|\cdot\|_1) \leqslant 1 - \gamma_1.$$

From this inequality we deduce that the set $\mathcal{A}$ is not closed on $(\mathcal{P}, \rho)$, since $\|\cdot\|_1 \in \mathcal{P} \setminus \mathcal{A}$ and for all $\epsilon > 0$ we can find $p \in B_\rho(\|\cdot\|_1, \epsilon)$ such that $p \in \mathcal{A}$ (recall that γ_1 can be chosen as close to one as we like).

Furthermore, from the instability result obtained in the previous section, we can deduce that $\mathcal{A}$ is not a ρ-open set either. In fact $|||\cdot||| \in \mathcal{A}$, $|||\cdot|||_{k_0} \notin \mathcal{A}$ and $\rho(|||\cdot|||, |||\cdot|||_{k_0}) \leqslant 1 - \gamma_{k_0}$, where $|||\cdot|||_{k_0}$ denotes the norm defined in last section and γ_{k_0} can again be chosen as close to 1 as we like.

By the equivalence among the metrics, $\mathcal{A}$ is neither closed nor open in $(\mathcal{P}, h)$ and the some holds for the set $\mathcal{A} \cap \mathcal{E}$ in the metric space $(\mathcal{E}, d)$.

On the other hand, notice that $\mathcal{P}$ is a convex cone, that is, if $p_1, p_2 \in \mathcal{P}$ and $\alpha_1, \alpha_2 \geqslant 0$, then $\alpha_1 p_1 + \alpha_2 p_2 \in \mathcal{P}$. Is the set $\mathcal{A}$ also a convex cone? From Theorem 2.1 we can deduce that $\mathcal{A}$ does contain rays. In fact, if $p(\cdot)$ is a norm in $\mathcal{P}$ satisfying property $(*)$ of Theorem 2.1, then $p(\cdot)$ is the starting point of an (open or closed) ray of norms contained in $\mathcal{A}$.

Let $C := \{(x_n) \in \ell_1:\ x_n \geqslant 0, \sum_{n=1}^\infty x_n = 1\}$. Of course, the set C is closed convex and bounded. If $S : \ell_1 \to \ell_1$ is the right shift mapping defined by $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$, then for every positive integer k, the set C is invariant under the mapping S^k. Moreover, S^k is fixed point free on C.

Therefore, every norm q equivalent to the standard one belongs to $\mathcal{P} \setminus \mathcal{A}$ whenever one of the mappings S^k is q-nonexpansive, that is, if for some positive integer k,

$$q(0, \overset{(k)}{\ldots}, 0, v_1, v_2, \ldots) \leqslant q(v_1, v_2, \ldots) \tag{12}$$

holds for every $v = (v_1, v_2, \ldots) \in \ell_1$. There are many well-known norms on ℓ_1 which satisfy condition (12) as, for instance, the following.

(1) The norm $\|\cdot\|_c$ defined by $\|(x_n)\|_c = \sum_{n=1}^\infty |x_n| + |\sum_{n=1}^\infty x_n|$.
(2) The family of norms defined by

$$\left\|(x_n)\right\|_\rho := \sum_{n=1}^\infty r_n|x_n|$$

where $\rho = (r_n)$ is a non-increasing sequence on the closed interval $[1, b]$ (for example, the $\|\cdot\|_1$-norm if $r_n = 1$ for all $n \in \mathbf{N}$).

(3) For a positive integer k and $a > 0$, let $\|\cdot\|_{a,k}$ be the norm defined on ℓ_1 as

$$\|x\|_{a,k} = a|x_1 + \cdots + x_k| + \|x\|_1.$$

For each $x \in \ell_1$ one has that $\|x\|_1 \leqslant \|x\|_{a,k} \leqslant (1 + a)\|x\|_1$, and it is obvious that

$$\big\|(0, \overset{(k)}{\ldots}, 0, v_1, v_2, \ldots)\big\|_{a,k} = \big\|(v_1, v_2, \ldots)\big\|_1 \leqslant \big\|(v_1, v_2, \ldots)\big\|_{a,k},$$

that is, S^k is $\|\cdot\|_{a,k}$-nonexpansive.

(4) The (dual) norm $\|x\|_l := \max\{\|x^+\|_1, \|x^-\|_1\}$, where x^+ and x^- stand respectively for the positive and the negative parts of $x = (x_n) \in \ell_1$. Notice that in [11] a fixed point free self-mapping of the weak*-compact convex set $K := \{(x_n) \in \ell_1\colon\ x_n \geqslant 0, \sum_{n=1}^{\infty} x_n \leqslant 1\}$ was given, namely $T(x) = (1 - \sum_{n=1}^{\infty} x_n, x_1, x_2, \ldots)$. This mapping T is 2 Lipschitzian with respect to the norm $\|\cdot\|_1$ but it is $\|\cdot\|_l$-nonexpansive on K.

(5) Of course, every positive linear combination of a finite number of norms satisfying condition (12) again satisfies such condition.

Notice that $\|\cdot\|_c$ and $\|\cdot\|_l$ fail to satisfy condition $(*)$ in Theorem 2.1.

The set of all of norms satisfying condition (12) is indeed large, but is far from be equal to the set $\mathcal{P} \setminus \mathcal{A}$: Let $U : \ell_1 \to \ell_1$ be the mapping defined by

$$U(x) = (0, x_1, 0, x_2, 0, x_3, 0, x_4, \ldots).$$

Of course, $\|U(x)\|_1 = \|x\|_1$, and U maps C into C. If $x = U(x)$ for some $x \in \ell_1$, then $x = 0_{\ell_1} \notin C$, and therefore U is a fixed point free self-mapping of C.

Let $\|\cdot\|_U$ be the norm on ℓ_1 defined by

$$\|x\|_U = \sum_{i=1}^{\infty} |x_{2i}| + \sum_{i=1}^{\infty} a|x_{2i-1}|$$

where $a > 1$ is previously given. It turns out that, for every $x \in \ell_1$

$$\big\|U(x)\big\|_U = \|x\|_1 \leqslant \|x\|_U$$

which implies that U is $\|\cdot\|_U$-nonexpansive on C. However, $\|\cdot\|_U$ fails (12), in spite that it belongs to $\mathcal{P} \setminus \mathcal{A}$.

We can define $\mathcal{N}_{U^k}$ as the set of all the equivalent renormings of ℓ_1 for which U^k is nonexpansive. Even more, for any operator V on ℓ_1 which leaves invariant the set C (that is, with $V(C) \subset C$) and which has no fixed points on C we can consider a subset $\mathcal{N}_V$ of $\mathcal{P} \setminus \mathcal{A}$, namely the set of those norms, say q, for which V is q-nonexpansive. In any case we have seen that the set $\mathcal{P} \setminus \mathcal{A}$ is, in some sense, quite complex.

These examples raise the problem (seemingly hard) of giving a characterization of the equivalent renormings of $(\ell_1, \|\cdot\|_1)$ lacking the FPP.

In reflexive Banach spaces, even in Hilbert spaces, is not known whether or not they can be equivalently renormed to fail the FPP. But in ℓ_1, since both sets $\mathcal{A}$ and $\mathcal{P} \setminus \mathcal{A}$ are nonempty, it raises as quite natural the question of *how large* is $\mathcal{A}$ (or $\mathcal{P} \setminus \mathcal{A}$) inside $\mathcal{P}$. For instance, is $\mathcal{A}$ (or $\mathcal{P} \setminus \mathcal{A}$) dense on $\mathcal{P}$ with respect to the previous metrics? Moreover, *'how large'* could also be stated in the sense of the categories, or in the sense of the different kinds of porosity. The results of the above section on stability as well as the above comments on $\mathcal{P} \setminus \mathcal{A}$ suggest that the set $\mathcal{A}$ should be negligible, or, in other words, that the property of failing FPP is generic for the elements of $\mathcal{P}$.

Acknowledgment

The authors would like to thank professor T. Domínguez Benavides for his helpful comments during the preparation of this paper.

References

[1] F.E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Natl. Acad. Sci. USA 54 (1965) 1041–1044.
[2] W.L. Bynum, Normal structure coefficients for normal structure for Banach spaces, Pacific J. Math. 86 (1980) 427–436.
[3] T. Domínguez-Benavides, Stability of the fixed point property for non-expansive mappings, Houston J. Math. 22 (4) (1996) 145–153.
[4] T. Domínguez Benavides, S. Phothi, Porosity of the fixed point property under renorming, in: Fixed Point Theory and Its Applications, Yokohama Publ., Yokohama, 2008, pp. 29–41.
[5] P.N. Dowling, P.K. Lin, B. Turrett, Direct sums of renormings of ℓ_1 and the fixed point property, Nonlinear Anal. 73 (2010) 591–599.
[6] H. Fetter, B. Gamboa de Buen, Banach spaces with a basis that are hereditarily asymptotically isometric to l_1 and the fixed point property, Nonlinear Anal. 71 (10) (2009) 4598–4608.
[7] D. Göhde, Zum Prinzip der kontraktiven Abbildung, Math. Nachr. 30 (1965) 251–258.

[8] C.A. Hernández-Linares, M.A. Japón, A renorming in some Banach spaces with applications to fixed point theory, J. Funct. Anal. 258 (2010) 3452–3468.
[9] W.A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72 (1964) 1004–1006.
[10] W.A. Kirk, B. Sims, Handbook of Metric Fixed Point Theory, Kluwer Academic Publishers, 2001.
[11] T.C. Lim, Asymptotic centers and nonexpansive mappings in conjugate Banach spaces, Pacific J. Math. 90 (1980) 135–143.
[12] P.K. Lin, There is an equivalent norm on ℓ_1 that has the fixed point property, Nonlinear Anal. 68 (8) (2008) 2303–2308.
[13] P.K. Lin, Renorming of ℓ_1 and the fixed point property, J. Math. Anal. Appl. 362 (2010) 534–541.
[14] E.M. Mazcuñán-Navarro, Stability of the fixed point property in Hilbert spaces, Proc. Amer. Math. Soc. 134 (2006) 129–138.
[15] S. Phothi, Genericity of the fixed point property under renorming, PhD thesis, Sevilla University, 2010.